Habilitation Thesis Reviewer's Report

Masaryk University	
Faculty	Faculty of Science
Procedure field	Algebra and number theory
Applicant	Bc. Lukáš Vokřínek, PhD
Applicant's home unit, institution	Department of Mathematics and Statistics, Faculty of Science MU
Habilitation thesis	Algorithmic aspects of topological problems
Reviewer	Dr. Jérôme Scherer
Reviewer's home unit, institution	Dpt de mathématiques, EPFL, Lausanne, Switzerland

The review text is attached in a separate pdf file.

Reviewer's questions for the habilitation thesis defence (number of questions up to the reviewer)

Six questions are attached in the same separate pdf file.

Conclusion

The habilitation thesis entitled "Algorithmic aspects of topological problems" by Lukáš Vokřínek ***fulfils*** the requirements expected of a habilitation thesis in the field of Algebraic Topology.

In Lausanne on 26 March 2018

REVIEW OF L. VOKŘÍNEK'S HABILITATION THESIS

J. SCHERER

The habilitation thesis submitted by L. Vokřínek consists in seven research articles dealing with questions sharing all the same main theme, namely that of the decidability of certain homotopy theoretic questions of classical nature, and if so, how to approach the solution from an algorithmic viewpoint. This collection of published work and preprints forms a very coherent ensemble focusing on an ambitious program. More precisely, the questions concern the computability of certain sets of homotopy classes $[X, Y]$, lifting and extension problems, determining whether two given maps are homotopic, etc.

1. Review

In this section I will review the content of the seven articles, highlighting the homotopical nature of Vokřínek work rather than the algorithms and programs.

A recurrent idea, which is present in the seven articles, is to find and choose, among a variety of tools and methods that have been studied over the years, those which are best suited to so-called *effective* computations, in the sense of Sergeraert. This means in particular that there is a way to program a computer to find a solution to certain problems I will mention below, but this is not to be confused with a way to find explicit answers to longstanding computational open problems. However, in the negative case, this often leads to proofs that certain questions are undecidable.

Hence, from the point of view of homotopy theory, it is not the originality of the methods that stands out, but rather the suitability of the choice of techniques. After all, a number of methods are available to attack a given problem and the authors have to pick the one that fits best to their computational objective. Of course certain theorems have to be adapted, even generalized, or analyzed more carefully than what had been done up till now to make sure that the statements correspond to the practical possibility to implement an actual algorithm.

The most prominent tool is the Postnikov tower of a space, that is used inductively to study maps *into a given space*. Hence the first step will always be related to Eilenberg-Mac Lane spaces, and k-invariants are then used to climb up the tower. The first paper is co-autored among others with Sergeraert and this is also the most cited of the seven papers. There, Vokřínek provides solide ground for computing homotopy classes of maps in the stable range, which includes in particular the case of stable homotopy groups of spheres. Subsequent papers develop methods to tackle more general, more complicated, and more delicate problems.

The second paper deals precisely with the k-invariants approach to decomposing a space, as well as a relative version. It turns out that the extension problem is undecidable for example when the target space is an even dimensional sphere, this is carefully studied in the third paper, where the author propose a clever reduction – and elementary at the same time – of a homotopy extension problem involving Whitehead products to an undecidable resolution of Diophantine equations. It also contains a simplicial version of Anick's #P-hardness result for 4-dimensional complexes.

In the fourth paper the methods become equivariant so as to solve a geometric embedding problem. Therefore effective computations must be performed equivariantly, but more new technology comes into play such as fiberwise homotopy theory (an ad hoc version of fiberwise H-spaces to be compared with the classical H-space structures used in the first article).

In paper number 5 Vokřínek, in a single author article, remarks that the generally undecidable extension problem studied in the third article is in fact decidable when the target is an *even* dimensional sphere, or more generally a space with finite torsion higher homotopy groups. The tendency to diversify the toolkit at hand is confirmed in the next paper where non-abelian polycylic groups appear, forcing the authors to extend the effective methods available for abelian groups. The question whether two maps are homotopic, or the computability of a set of homotopy classes of maps, becomes unstable as the only restriction on the target is simple connectivity for the former question, and the additional assumption that the source be a suspension is imposed for the latter. The final paper could be considered as an algebraic appendix to the article 4, but it has of course its own interest and separate applications developed intrinsically.

To sum up, Vokřínek has spent a great amount of work on a set of related problems, originating actually in general questions asked at the beginning of the first paper. In the following papers the answers become more precise, new methods are introduced so as to provide more general answers. This makes for a coherent program that does not follow current (and passing?) trends, but reflects Vokřínek's own interests and priorities.

2. Reviewer's questions for the habilitation thesis defence

(1) I wonder how the work on embedding of manifolds in Euclidean space relates to known results, for example Don Davis' embedding theorems for real projective spaces. Often certain bounds are known but the precise best dimension is not. Do the hardness or the undecidability shed some light on this kind of theorems?

(2) While reading the articles of this habilitation thesis I felt that there is a relationship with Jesper Møller's work on lifting and extensions, in particular the Pacific Journal paper "Spaces of sections of Eilenberg-Mac Lane fibrations". I do not have a precise question, but am rather just wondering if this article is relevant.

(3) One of the difficulties the author faces is that the computation of homotopy groups of spheres for example has been attacked with powerful and ingenious methods, therefore one should not expect to be able to do better by "brute force". Nevertheless there are other areas of algebraic topology, more recent ones also, where even elementary computations are new and interesting. Could one imagine to apply the effective approach to such questions, for example those related to topological complexity?

(4) What does it mean for $K(\mathbb{Z}, 1)$ to have polynomial time homology? My impression is that it is more than saying that the homology of a circle can be computed by a machine (after all one could just tell the computer that $K(Z, 1)$ is a Moore space), but it probably has to do with the inductive approach?

(5) Is there a link with Cartan and Serre's computation of the homology of Eilenberg-Mac Lane spaces? Alain Clément for example has implemented their description in a (rudimentary) program 15 years ago. I wonder if a mixture of techniques, allowing the effective algorithm to access certain elements of informations coming for example from known spectral sequence calculations, could speed up the computations? Likewise, since the first homotopy groups of spheres are actually known, could this be used to start the induction higher up to try to have access to new information instead of trying to do all computations from scratch? Of course in the early stages of the project it is important to validate the methods by comparing the results to known propositions.

(6) I know that this computation is theoretically outside of the range studied in the thesis, but for personal reasons I would be interested in knowing who is $[S^9, \Sigma^2\mathbb{C}P^2]^{C_2}$, the set of homotopy classes of equivariant maps from the unit sphere in $\mathbb{C}^5$, with conjugation action, to the smash product of $\mathbb{C}P^2$ with a circle with trivial action and one circle with sign action.